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Irreducible characters and normal subgroups in groups of odd order

A. Laradji

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

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ABSTRACT

If b is a p-block of a normal subgroup N of a finite group G of odd order and b^* is its Brauer correspondent in $N_N(Q)$, where Q is a defect group of b, then for any p-block B of G over b, there exists a natural height-preserving bijection from the set of irreducible complex characters of B lying over height-zero characters onto the set of irreducible complex characters of the Harris–Knörr correspondent B^* of B over b^* lying over height-zero characters.

1. Introduction

Let p be a prime number and let G be a finite p-solvable group. In [13], to every ordinary character $\chi \in \mathrm{Irr}(G)$, G. Navarro associates a canonical pair (Q, δ), where Q is a p-subgroup of G and $\delta \in \mathrm{Irr}(Q)$. This pair, which is uniquely determined by χ up to G-conjugacy, is called a vertex of χ.

If Q is a p-subgroup of G and δ is an arbitrary irreducible character of Q, then in accordance with [13], we denote by $\mathrm{Irr}(G|Q, \delta)$, the set of irreducible characters of G having vertex (Q, δ). The union of the sets $\mathrm{Irr}(G|Q, \delta)$, with δ running through $\mathrm{Irr}(Q)$, is denoted by $\mathrm{Irr}(G|Q)$.

Now assume that $|G|$ is odd. Next fix a p-subgroup Q of G. For any $\delta \in \mathrm{Irr}(Q)$, Navarro defined, in [14], a natural injection from $\mathrm{Irr}(G|Q, \delta)$ into $\mathrm{Irr}(N_G(Q, \delta)|\delta)$, the set of irreducible characters of the stabilizer $N_G(Q, \delta)$ of (Q, δ) in G that lie over δ. (See [14, Theorem 5.4].) By Clifford's theory, the composition of this injection with the induction map defines a natural injection $\Omega_{G,\delta}$ from $\mathrm{Irr}(G|Q, \delta)$

E-mail address: alaradji@ksu.edu.sa.

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into $\mathrm{Irr}(N_G(Q)|\delta)$. The $\Omega_{G,\delta}$'s taken together give rise to a natural injection $\Omega_{G,Q}$ from $\mathrm{Irr}(G|Q)$ into $\mathrm{Irr}(N_G(Q))$ (see Theorem 4.1 below). Our first main result is:

Theorem A. *Let $N \lhd G$, where G is a group of odd order and let $\mu \in \mathrm{Irr}(N|Q)$ where Q is a p-subgroup of N. Let $\widetilde{\mu} = \Omega_{N,Q}(\mu)$. Then there exists a natural p-defect-preserving bijection from the set $\mathrm{Irr}(G|\mu)$ of irreducible characters of G lying over μ onto the set $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$ of irreducible characters of $N_G(Q)$ lying over $\widetilde{\mu}$.*

Let b be a p-block of a normal subgroup N of G. Suppose Q is a defect group of b, and let $\widetilde{b}$ be the Brauer correspondent of b in $N_N(Q)$. By a well-known result of M. Harris and R. Knörr [4], the Brauer correspondence gives a defect group preserving bijection from the set $\mathrm{Bl}(N_G(Q)|\widetilde{b})$ of p-blocks of $N_G(Q)$ that cover $\widetilde{b}$ onto the set $\mathrm{Bl}(G|b)$ of p-blocks of G that cover b.

Denote by $\mathrm{Irr}_0(b)$, the set of irreducible characters in b of height zero. Next, for a p-block $B \in \mathrm{Bl}(G|b)$, let

$$\mathrm{Irr}\big(B|\mathrm{Irr}_0(b)\big) = \big\{\chi \in \mathrm{Irr}(B)\colon\ \chi \text{ lies over some } \mu \in \mathrm{Irr}_0(b)\big\}.$$

Now the following is a consequence of Theorem A.

Theorem B. *Let N be a normal subgroup of a group G of odd order, and let B and b be p-blocks of G and N respectively such that B covers b. Let Q be a defect group of b and write $\widetilde{b}$ for the Brauer correspondent of b in $N_N(Q)$. If $\widetilde{B} \in \mathrm{Bl}(N_G(Q)|\widetilde{b})$ is the Harris–Knörr correspondent of B, then there exists a natural height-preserving bijection from $\mathrm{Irr}(B|\mathrm{Irr}_0(b))$ onto $\mathrm{Irr}(\widetilde{B}|\mathrm{Irr}_0(\widetilde{b}))$.*

If, in Theorem B, Q is abelian, then by a result of P. Fong [2], every character in $\mathrm{Irr}(b)$ or in $\mathrm{Irr}(\widetilde{b})$ is of height zero. So, in this case, Theorem B asserts the existence of a natural one-to-one height-preserving correspondence between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(\widetilde{B})$.

Theorem B implies, in particular, that B and $\widetilde{B}$ contain equal numbers of irreducible characters over height-zero characters. This fact is the ordinary analogue, for groups of odd order, of the main theorem of [9].

We obtain Theorem B as a consequence of Theorem A. The argument used to prove Theorem A relies heavily on a theorem of I.M. Isaacs [5, Theorem 10.9]. This result of Isaacs provides a natural correspondence of characters in solvable groups of odd order. We should emphasize that the argument used by Isaacs to construct this correspondence only applies for solvable groups with a certain oddness condition (the argument depends on some deep properties of the odd fully ramified sections). This is one major reason we are only able to prove our results for solvable groups of odd order.

We do not know to what extent beyond groups of odd order Theorem B remains true. However, we suspect for all p-solvable groups that a weaker statement than Theorem B is true: that Harris–Knörr correspondents always contain equal numbers of irreducible characters over height-zero characters.

2. Isaacs' character correspondence

Throughout this section, a prime p and a finite group G of odd order are fixed. Let H be any subgroup of G. Denote by $\mathrm{Irr}_{p'}(H)$, the set of irreducible characters of H of degree not divisible by p. Let $P \in \mathrm{Syl}_p(H)$. In [5], I.M. Isaacs discovered a natural bijection $F^H_{N_H(P)}$ from $\mathrm{Irr}_{p'}(H)$ onto $\mathrm{Irr}_{p'}(N_H(P))$ (see [5, Theorem 10.9]). One property of this correspondence, we shall use, is the fact that $F^{H^x}_{N_{H^x}(P^x)}(\chi^x) = F^H_{N_H(P)}(\chi)^x$, for any $\chi \in \mathrm{Irr}_{p'}(H)$ and any $x \in G$. Another important property of the bijection concerns Gajendragadkar's special characters [3]. (We assume that the reader is familiar with the definition and basic properties of these characters.) Let $\mathcal{X}_{p'}(H)$ be the set of p'-special characters of H. Then $\mathcal{X}_{p'}(H) \subseteq \mathrm{Irr}_{p'}(H)$ and Corollary 3.3 in [11] says that $F^H_{N_H(P)}$ maps $\mathcal{X}_{p'}(H)$ onto $\mathcal{X}_{p'}(N_H(P))$.

Let K be a subgroup of G such that $N_H(P) \subseteq K \subseteq H$. Then $P \in \mathrm{Syl}_p(K)$ and $N_K(P) = N_H(P)$. Now consider the bijection $(F^K_{N_K(P)})^{-1} \circ F^H_{N_H(P)}$ from $\mathrm{Irr}_{p'}(H)$ onto $\mathrm{Irr}_{p'}(K)$. Suppose that R is another Sylow p-subgroup of H with $N_H(R) \subseteq K$. Then $R \in \mathrm{Syl}_p(K)$ and thus $R = P^k$ for some $k \in K$. Let $\chi \in \mathrm{Irr}_{p'}(H)$. Then $F^H_{N_H(R)}(\chi) = F^{H^k}_{N_{H^k}(P^k)}(\chi^k) = F^H_{N_H(P)}(\chi)^k$, and so

$$\begin{aligned}\big(F^K_{N_K(R)}\big)^{-1}\big(F^H_{N_H(R)}(\chi)\big) &= \big[\big(F^K_{N_K(P)}\big)^{-1}\big(F^H_{N_H(P)}(\chi)\big)\big]^k\\ &= \big(F^K_{N_K(P)}\big)^{-1}\big(F^H_{N_H(P)}(\chi)\big).\end{aligned}$$

Hence the bijection $(F^K_{N_K(P)})^{-1} \circ F^H_{N_H(P)}$ does not depend on the choice of the Sylow p-subgroup P of H. We may then denote this map by F^H_K.

We record a few properties of the map F^H_K in the following lemma.

Lemma 2.1. *Let H and K be subgroups of G such that $N_H(P) \subseteq K \subseteq H$, where $P \in \mathrm{Syl}_p(H)$.*

(i) $F^{H^x}_{K^x}(\chi^x) = F^H_K(\chi)^x$, *for any $\chi \in \mathrm{Irr}_{p'}(H)$ and any $x \in G$.*
(ii) F^H_K *restricts to a bijection of $\mathcal{X}_{p'}(H)$ onto $\mathcal{X}_{p'}(K)$.*
(iii) *Let H_0 be a subgroup of H containing P and let $K_0 = K \cap H_0$. Then $N_{H_0}(P) \subseteq K_0$, and if $\chi_0 \in \mathrm{Irr}_{p'}(H_0)$ is such that $\chi_0^H \in \mathrm{Irr}(H)$ and $F^{H_0}_{K_0}(\chi_0)^K \in \mathrm{Irr}(K)$, then $F^H_K(\chi_0^H) = F^{H_0}_{K_0}(\chi_0)^K$.*

Proof. (i) For $\chi \in \mathrm{Irr}_{p'}(H)$ and $x \in G$, we have

$$\begin{aligned}F^{H^x}_{K^x}\big(\chi^x\big) &= \big(F^{K^x}_{N_{K^x}(P^x)}\big)^{-1}\big(F^{H^x}_{N_{H^x}(P^x)}\big(\chi^x\big)\big)\\ &= \big(F^{K^x}_{N_{K^x}(P^x)}\big)^{-1}\big(F^H_{N_H(P)}(\chi)^x\big)\\ &= \big[\big(F^K_{N_K(P)}\big)^{-1}\big(F^H_{N_H(P)}(\chi)\big)\big]^x\\ &= F^H_K(\chi)^x.\end{aligned}$$

(ii) Since $F^H_{N_H(P)}$ restricts to a bijection of $\mathcal{X}_{p'}(H)$ onto $\mathcal{X}_{p'}(N_H(P))$ and $(F^K_{N_K(P)})^{-1}$ restricts to a bijection from $\mathcal{X}_{p'}(N_H(P))$ onto $\mathcal{X}_{p'}(K)$, we have the result.

(iii) First $N_{H_0}(P) = H_0 \cap N_H(P) \subseteq H_0 \cap K = K_0$. Next by [11, Theorem 3.5], we have $F^H_{N_H(P)}(\chi_0^H) = F^{H_0}_{N_{H_0}(P)}(\chi_0)^{N_H(P)}$ and $F^K_{N_K(P)}(F^{H_0}_{K_0}(\chi_0)^K) = [F^{K_0}_{N_{K_0}(P)}(F^{H_0}_{K_0}(\chi_0))]^{N_K(P)}$. Hence, (using the fact that $N_{K_0}(P) = N_{H_0}(P)$ and $N_K(P) = N_H(P)$),

$$\begin{aligned}F^K_{N_K(P)}\big(F^{H_0}_{K_0}(\chi_0)^K\big) &= F^{H_0}_{N_{H_0}(P)}(\chi_0)^{N_K(P)}\\ &= F^H_{N_H(P)}\big(\chi_0^H\big).\end{aligned}$$

Then $F^H_K(\chi_0^H) = (F^K_{N_K(P)})^{-1}(F^H_{N_H(P)}(\chi_0^H)) = F^{H_0}_{K_0}(\chi_0)^K$, as needed. $\square$

Suppose $P \in \mathrm{Syl}_p(G)$ and Q is a p-subgroup of G such that $N_G(P) \subseteq N_G(Q)$. Let $\chi \in \mathrm{Irr}_{p'}(G)$. Then χ belongs to a p-block B of G with defect group P. By [10, Theorem 5.3.8], there exists a unique p-block $\widetilde{B}$ of $N_G(Q)$ having P as a defect group and such that $B = \widetilde{B}^G$. Now the main result of this section is the following.

Proposition 2.2. *Let $P \in \mathrm{Syl}_p(G)$ and let Q be a p-subgroup of G such that $N_G(P) \subseteq N_G(Q)$. Let $\chi \in \mathrm{Irr}_{p'}(G)$ and $\widetilde{\chi} = F^{G}_{N_G(Q)}(\chi)$. If B is the p-block of G to which χ belongs, then $\widetilde{\chi}$ belongs to the unique p-block $\widetilde{B}$ of $N_G(Q)$ having defect group P and such that $\widetilde{B}^G = B$.*

Before proving this proposition, we need a lemma.

Lemma 2.3. *Let $P \in \mathrm{Syl}_p(G)$ and let Q be a p-subgroup of G such that $N_G(P) \subseteq N_G(Q)$. Suppose L is a normal p'-subgroup of G and write $\widetilde{L} = L \cap N_G(Q)$, so that $\widetilde{L} = C_L(Q)$. Let $\nu \in \mathrm{Irr}(L)$ be G-invariant and let $\widetilde{\nu} \in \mathrm{Irr}(\widetilde{L})$ be the Glauberman correspondent of ν with respect to the action of Q on L. If $\chi \in \mathrm{Irr}_{p'}(G)$ and $\widetilde{\chi} = F^{G}_{N_G(Q)}(\chi)$, then χ lies over ν if and only if $\widetilde{\chi}$ lies over $\widetilde{\nu}$.*

Proof. Since ν is invariant in G, then $\widetilde{\nu}$ is invariant in $N_G(Q)$. Next as $P \subseteq N_G(Q)$, then P acts on $\widetilde{L}$ and $\widetilde{\nu} \in \mathrm{Irr}_P(\widetilde{L})$. Let $M = L \cap N_G(P)$. Then $M = C_L(P)$, and

$$C_{\widetilde{L}}(P) = \widetilde{L} \cap N_G(P) = L \cap N_G(P) = M.$$

Now let $\xi \in \mathrm{Irr}(M)$ be the Glauberman correspondent of $\widetilde{\nu}$ with respect to the action of P on $\widetilde{L}$.

By [6, Theorem 13.1(c)], $\widetilde{\nu}$ is the unique irreducible constituent of $\nu_{\widetilde{L}}$ satisfying $p \nmid [\nu_{\widetilde{L}}, \widetilde{\nu}]$ and ξ is the unique irreducible constituent of $\widetilde{\nu}_M$ such that $[\widetilde{\nu}_M, \xi]$ is not divisible by p. Therefore ξ is the unique irreducible constituent of ν_M such that $[\nu_M, \xi]$ is not divisible by p. It follows by [6, Theorem 13.1(c)] that ξ is the Glauberman correspondent of ν with respect to the action of P on L.

Let $\theta = F^{G}_{N_G(P)}(\chi)$. Then $\theta = F^{N_G(Q)}_{N_G(P)}(\widetilde{\chi})$. By [14, Corollary 3.4], χ lies over ν if and only if θ lies over ξ, and $\widetilde{\chi}$ lies over $\widetilde{\nu}$ if and only if θ lies over ξ. The result is then immediate. $\square$

Let B be a p-block of an arbitrary (finite) p-solvable group E and let $L \lhd E$ be a p'-group. Then there is a (uniquely determined up to E-conjugacy) character $\nu \in \mathrm{Irr}(L)$ such that all members of $\mathrm{Irr}(B)$ lie over ν. Accordingly, ν is said to lie under B. In case $L = O_{p'}(E)$ and ν is invariant in E, it turns out that $\mathrm{Irr}(B)$ is precisely the set $\mathrm{Irr}(E|\nu)$ of irreducible characters of E that lie over ν. (See, for instance, [12, Theorem 10.20].)

Proof of Proposition 2.2. Let $L = O_{p'}(G)$ and $\widetilde{L} = L \cap N_G(Q)$. Then $\widetilde{L} = C_L(Q)$. Also by [16, Lemma 3.7], $\widetilde{L} = O_{p'}(N_G(Q))$. Now choose $\xi \in \mathrm{Irr}(\widetilde{L})$ under $\widetilde{B}$. Write $\widetilde{T}$ for the inertial group of ξ in $N_G(Q)$ and let $\widetilde{b}$ be the block of $\widetilde{L}$ to which ξ belongs. Then $\mathrm{Irr}(\widetilde{b}) = \{\xi\}$, so that $\widetilde{T}$ is the stabilizer of $\widetilde{b}$ in $N_G(Q)$. We have $\widetilde{B} \in \mathrm{Bl}(N_G(Q)|\widetilde{b})$ and we let $\widetilde{B}' \in \mathrm{Bl}(\widetilde{T}|\widetilde{b})$ be the Fong–Reynolds correspondent of $\widetilde{B}$. By [10, Theorem 5.5.10(iv)], as $\widetilde{B}$ covers all $N_G(Q)$-conjugates of $\widetilde{b}$ and P is a defect group for $\widetilde{B}$, we may assume that $\widetilde{B}'$ has defect group P.

Next, let $\nu \in \mathrm{Irr}_Q(L)$ correspond to ξ under the Glauberman correspondence with respect to the action of Q on L. By the discussion in the beginning of Section 3 of [16] and Theorem 3.2 of the same paper, we note that ν lies under B. Now let T be the inertial group of ν in G. Then $\widetilde{T} = T \cap N_G(Q)$.

Suppose $T = G$. Then ξ is $N_G(Q)$-invariant, and so $\mathrm{Irr}(\widetilde{B}) = \mathrm{Irr}(N_G(Q)|\xi)$. Since $\chi \in \mathrm{Irr}(B)$, χ lies over ν. Then by Lemma 2.3, $\widetilde{\chi}$ lies over ξ and hence $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{B})$. We clearly have the result in this case.

Suppose now that $T < G$. Since ν is Q-invariant, we have $Q \subseteq T$. Then $\widetilde{T} = N_T(Q)$ and hence $(\widetilde{B}')^T$ is defined. Let b be the block of L to which ν belongs. Then $\mathrm{Irr}(b) = \{\nu\}$, so that T is the stabilizer of b in G. As ν lies under B, then $B \in \mathrm{Bl}(G|b)$. Now let $B' \in \mathrm{Bl}(T|b)$ be the Fong–Reynolds correspondent of B. Then ν lies under B', and so by Theorem 4.6 in [16] we have $B' = (\widetilde{B}')^T$.

Next as $P \subseteq \widetilde{T}$, we have $P \subseteq T$. Then

$$N_T(P) = T \cap N_G(P) \subseteq T \cap N_G(Q) = N_T(Q).$$

Let $\psi \in \mathrm{Irr}(T|\nu)$ be the Clifford correspondent of χ. Then $\psi \in \mathrm{Irr}_{p'}(T)$ (as $\chi \in \mathrm{Irr}_{p'}(G)$), and ψ belongs to a block of T covering b. As $\psi^G = \chi \in \mathrm{Irr}(B)$, Theorem 5.5.10 of [10] implies that $\psi \in \mathrm{Irr}(B')$. Now let $\widetilde{\psi} = F^T_{N_T(Q)}(\psi)$. Since $T < G$, then by induction, $\widetilde{\psi}$ belongs to $\widetilde{B}'$. Then by [10, Theorem 5.5.10(ii)] again, the irreducible character $\widetilde{\psi}^{N_G(Q)}$ belongs to $\widetilde{B}$. Now, using Lemma 2.1(iii), we get

$$\widetilde{\chi} = F^G_{N_G(Q)}(\chi) = F^G_{N_G(Q)}\big(\psi^G\big) = F^T_{N_T(Q)}(\psi)^{N_G(Q)} = \widetilde{\psi}^{N_G(Q)},$$

and so $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{B})$, as wanted. $\square$

3. Navarro's vertices

Let G be a p-solvable group. Suppose $\alpha \in \mathcal{X}_p(G)$, the set of p-special characters of G, and $\beta \in \mathcal{X}_{p'}(G)$. It turns out that the character $\alpha\beta$ is irreducible and that this factorization is unique (see [3, Proposition 7.1]). Any irreducible character χ of G which can be expressed in this factored manner is called a p-factorable character, and we write χ_p and $\chi_{p'}$ for the p-special and p'-special factors, respectively.

A character pair in G is a pair (H,θ) where H is a subgroup of G and $\theta \in \mathrm{Irr}(H)$. For character pairs (H,θ) and (K,ξ), write $(H,\theta) \leqslant (K,\xi)$ if $H \subseteq K$ and θ is a constituent of ξ_H. This defines a partial order on the set of character pairs of G. Let G act on this set by $(H,\theta)^g = (H^g,\theta^g)$, and note that this action respects the partial order.

A character pair (N,θ) in which $N \lhd G$ and θ is p-factorable is called a p-factorable normal pair. We write $\mathcal{E}(G)$ for the set of all p-factorable normal pairs in G, and the set of maximal elements of $\mathcal{E}(G)$ is denoted by $\mathcal{E}^*(G)$.

Let $\chi \in \mathrm{Irr}(G)$. It is shown in [1] that there exists $(L,\rho) \in \mathcal{E}^*(G)$ with $(L,\rho) \leqslant (G,\chi)$ and that for any $(N,\theta) \in \mathcal{E}(G)$ such that $(N,\theta) \leqslant (G,\chi)$, there exists an element $g \in G$ for which $(N,\theta)^g \leqslant (L,\rho)$. It is then plain that the pair (L,ρ) is uniquely determined up to G-conjugacy. Any such pair is called a maximal p-factorable normal pair under χ.

In [13], Navarro defines the normal nucleus (W,γ) of a character $\chi \in \mathrm{Irr}(G)$ as follows. If χ is p-factorable, then we let $(W,\gamma) = (G,\chi)$. If χ is not p-factorable, choose a maximal p-factorable normal pair (L,ρ) under χ. Then, as we have seen above, (L,ρ) is unique up to G-conjugacy. Let I be the inertial group of ρ in G. Then, in view of Corollary 2.4 of [13], $I < G$. If $\psi \in \mathrm{Irr}(I|\rho)$ is the Clifford correspondent of χ, then ψ is uniquely determined by χ up to G-conjugacy. Now recursively define the normal nucleus of χ to be any G-conjugate of any normal nucleus of ψ.

It follows from the above construction that the set of normal nuclei of $\chi \in \mathrm{Irr}(G)$ is a full G-conjugacy class of character pairs. Also, for any normal nucleus (W,γ) of χ, note that γ is p-factorable and $\chi = \gamma^G$.

Let $\chi \in \mathrm{Irr}(G)$ and suppose (W,γ) is a normal nucleus for χ. If Q is a Sylow p-subgroup of W, then $\delta = (\gamma_p)_Q$ is an irreducible character of Q by [3, Proposition 6.1]. According to [13], any pair (Q,δ) obtained in this way is called a vertex for χ. The set of vertices of χ turns out to be a G-conjugacy class of pairs. (See Section 3 in [13].)

4. Navarro's character injection

In this section we let G be a group of odd order and Q, a p-subgroup of G. If $\delta \in \mathrm{Irr}(Q)$, we write $\mathrm{Irr}(G|Q,\delta)$ for the set of irreducible characters χ having vertex (Q,δ). Also, we denote by $\mathrm{Irr}(G|Q)$, the union of the character sets $\mathrm{Irr}(G|Q,\delta)$, with δ running through $\mathrm{Irr}(Q)$.

Let $\delta \in \mathrm{Irr}(Q)$ and suppose $\chi \in \mathrm{Irr}(G|Q,\delta)$. Then χ has a normal nucleus (W,γ) with $Q \in \mathrm{Syl}_p(W)$ and $\delta = (\gamma_p)_Q$. Let $\widetilde{\gamma_p} = (\gamma_p)_{N_W(Q)}$ and $\widetilde{\gamma_{p'}} = F^W_{N_W(Q)}(\gamma_{p'})$. Then $\widetilde{\gamma_p} \in \mathcal{X}_p(N_W(Q))$ by [3, Proposition 6.1], and $\widetilde{\gamma_{p'}} \in \mathcal{X}_{p'}(N_W(Q))$. Also, as $(\gamma_p)_Q = \delta$, note that $N_W(Q) = N_W(Q,\delta)$.

In [14], Navarro has shown that the induced character $(\widetilde{\gamma_p}\widetilde{\gamma_{p'}})^{N_G(Q,\delta)}$ is uniquely defined by χ (and does not depend on the choice of the nucleus (W,γ)), and that the correspondence $\chi \mapsto (\widetilde{\gamma_p}\widetilde{\gamma_{p'}})^{N_G(Q,\delta)}$ defines a (natural) injection $\Psi_{G,\delta}$ from $\mathrm{Irr}(G|Q,\delta)$ into $\mathrm{Irr}(N_G(Q,\delta)|\delta)$. (See [14, Theorem 5.4].) Next, by Clifford theory [6, Theorem 6.11], the correspondence $\chi \mapsto \Psi_{G,\delta}(\chi)^{N_G(Q)}$ defines a natural injection $\Omega_{G,\delta}$ from $\mathrm{Irr}(G|Q,\delta)$ into $\mathrm{Irr}(N_G(Q)|\delta)$.

Suppose now that $\chi \in \mathrm{Irr}(G|Q, \delta')$ for another $\delta' \in \mathrm{Irr}(Q)$. Then there is a normal nucleus (W', γ') for χ such that $Q \in \mathrm{Syl}_p(W')$ and $\delta' = (\gamma'_p)_Q$. As nuclei of χ are G-conjugate, we have $(W', \gamma') = (W^g, \gamma^g)$ for some $g \in G$. Now both Q and Q^g are Sylow p-subgroups of W^g. Then, by Sylow theory, there exists $w \in W$ for which $Q = Q^{wg}$. So, in particular, $wg \in N_G(Q)$. Now

$$\begin{aligned}(\gamma'_p)_{N_{W'}(Q)} &= \bigl((\gamma^{wg})_p\bigr)_{N_{W^{wg}}(Q)}\\ &= \bigl((\gamma_p)_{N_W(Q)}\bigr)^{wg}\\ &= (\widetilde{\gamma_p})^{wg}.\end{aligned}$$

Also

$$\begin{aligned}F^{W'}_{N_{W'}(Q)}(\gamma'_{p'}) &= F^{W^{wg}}_{N_{W^{wg}}(Q)}\bigl((\gamma^{wg})_{p'}\bigr)\\ &= F^{W}_{N_W(Q)}(\gamma_{p'})^{wg}\\ &= (\widetilde{\gamma_{p'}})^{wg}.\end{aligned}$$

Then

$$\begin{aligned}\Omega_{G,\delta'}(\chi) &= \bigl((\gamma'_p)_{N_{W'}(Q)} F^{W'}_{N_{W'}(Q)}(\gamma'_{p'})\bigr)^{N_G(Q)}\\ &= \bigl((\widetilde{\gamma_p}\widetilde{\gamma_{p'}})^{wg}\bigr)^{N_G(Q)}\\ &= \Omega_{G,\delta}(\chi),\end{aligned}$$

where the last equality follows from the fact that $wg \in N_G(Q)$. Therefore the $\Omega_{G,\delta}$'s taken together define a natural map $\Omega_{G,Q}$ from $\mathrm{Irr}(G|Q)$ to $\mathrm{Irr}(N_G(Q))$.

Theorem 4.1. *$\Omega_{G,Q}$ is a natural injection from* $\mathrm{Irr}(G|Q)$ *into* $\mathrm{Irr}(N_G(Q))$.

Proof. Let $\chi, \theta \in \mathrm{Irr}(G|Q)$ and suppose $\Omega_{G,Q}(\chi) = \Omega_{G,Q}(\theta)$. Then $\chi \in \mathrm{Irr}(G|Q, \delta)$ and $\theta \in \mathrm{Irr}(G|Q, \varepsilon)$ for some $\delta, \varepsilon \in \mathrm{Irr}(Q)$. Now both δ and ε lie under $\Omega_{G,Q}(\chi)$. Hence $\varepsilon = \delta^x$ for some $x \in N_G(Q)$, and so $(Q, \varepsilon) = (Q, \delta)^x$. It follows that $\theta \in \mathrm{Irr}(G|Q, \delta)$. Now $\Omega_{G,\delta}(\theta) = \Omega_{G,Q}(\theta) = \Omega_{G,\delta}(\chi)$. But we know that $\Omega_{G,\delta}$ is one-to-one. Hence $\theta = \chi$. The map $\Omega_{G,Q}$ is then one-to-one. □

The next result is needed in Section 5.

Lemma 4.2. *Let $N \lhd G$ with $Q \subseteq N$, and suppose $\mu \in \mathrm{Irr}(N|Q)$. Let $\widetilde{\mu} = \Omega_{N,Q}(\mu)$. If $x \in N_G(Q)$, then $\mu^x \in \mathrm{Irr}(N|Q)$ and $\widetilde{\mu}^x = \Omega_{N,Q}(\mu^x)$.*

Proof. Let (Q, δ) be a vertex of μ. Then there is a normal nucleus (W, γ) for μ such that $Q \in \mathrm{Syl}_p(W)$ and $\delta = (\gamma_p)_Q$. Now $\widetilde{\mu} = ((\gamma_p)_{N_W(Q)} F^W_{N_W(Q)}(\gamma_{p'}))^{N_N(Q)}$.

Let $x \in N_G(Q)$. Then, by the definition of the normal nucleus, it is easy to see that (W^x, γ^x) is a nucleus for μ^x. Now Q $(= Q^x) \in \mathrm{Syl}_p(W^x)$ and $((\gamma^x)_p)_Q = ((\gamma_p)_Q)^x = \delta^x$. Then (Q, δ^x) is a vertex of μ^x and so, in particular, $\mu^x \in \mathrm{Irr}(N|Q)$.

Now $\Omega_{N,Q}(\mu^x) = [((\gamma^x)_p)_{N_{W^x}(Q)} F^{W^x}_{N_{W^x}(Q)}((\gamma^x)_{p'})]^{N_N(Q)}$. But $((\gamma^x)_p)_{N_{W^x}(Q)} = ((\gamma_p)_{N_W(Q)})^x$ and $F^{W^x}_{N_{W^x}(Q)}((\gamma^x)_{p'}) = F^W_{N_W(Q)}(\gamma_{p'})^x$. It follows that $\Omega_{N,Q}(\mu^x) = \widetilde{\mu}^x$. The proof is now complete. □

We now turn our attention to blocks. Let B be a p-block of a p-solvable group E. Write $\mathrm{Irr}_0(B)$ for the set of characters in $\mathrm{Irr}(B)$ of height zero. Suppose D is a defect group of B, and let $\widetilde{B}$ be the Brauer correspondent of B in $N_E(D)$. A theorem of T. Okuyama and M. Wajima [15] asserts that $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(\widetilde{B})|$, thus establishing the Alperin–McKay conjecture for p-solvable groups. Our next result shows that when E has odd order, a natural one-to-one correspondence exists between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(\widetilde{B})$.

Theorem 4.3. *Let B be a p-block of G. Suppose D is a defect group for B and write $\widetilde{B}$ for the Brauer correspondent of B in $N_G(D)$. Then every $\chi \in \mathrm{Irr}_0(B)$ lies in $\mathrm{Irr}(G|D)$. Furthermore, $\Omega_{G,D}$ restricts to a bijection from $\mathrm{Irr}_0(B)$ onto $\mathrm{Irr}_0(\widetilde{B})$.*

Before proving this theorem, we need an easy lemma.

Lemma 4.4. *Let E be a p-solvable group and let $\alpha, \beta \in \mathrm{Irr}(E)$ with α p-special and β p'-special. Then β and $\alpha\beta$ belong to the same p-block of E.*

Proof. Let $N = O_{p'}(E)$. The irreducible constituents of α_N, being p-special by [7, Lemma 2.2], must all be 1_N. Then α belongs to the principal block of E. Now the result follows by [16, Lemma 2.9]. $\square$

We can now prove Theorem 4.3.

Proof of Theorem 4.3. Let $\chi \in \mathrm{Irr}_0(B)$. First we prove that χ has vertex (D, δ) for some $\delta \in \mathrm{Irr}(D)$.

Suppose that (Q, ε) is a vertex of χ. Then there exists a normal nucleus (W, γ) for χ such that $Q \in \mathrm{Syl}_p(W)$ and $\varepsilon = (\gamma_p)_Q$. Since γ is p-factorable, Lemma 2.10 of [16] tells us that γ belongs to a block b of W with defect group Q. Now, as $\chi = \gamma^G$, we have that b^G is defined and $B = b^G$ by [10, Lemma 5.3.1]. Then by [10, Lemma 5.3.3], $Q^g \subseteq D$ for some $g \in G$. We have $\chi(1)_p = |G : Q|_p \gamma(1)_p$. On the other hand, since $\chi \in \mathrm{Irr}_0(B)$, $\chi(1)_p = |G : D|_p$. We are then forced to have $Q^g = D$ and $\gamma(1)_p = 1$. Then, by letting $\delta = \varepsilon^g$, the character χ has vertex (D, δ). Now, for our purposes, there is no loss in assuming that $(D, \delta) = (Q, \varepsilon)$.

By Theorem 4.1, we have a natural injection $\Omega_{G,D}$ from $\mathrm{Irr}(G|D)$ to $\mathrm{Irr}(N_G(D))$. Now to finish the proof, since $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(\widetilde{B})|$, it suffices to show that $\Omega_{G,D}(\mathrm{Irr}_0(B)) \subseteq \mathrm{Irr}_0(\widetilde{B})$.

By Lemma 4.4, we have $\gamma_{p'} \in \mathrm{Irr}(b)$. Now in view of Proposition 2.2, the character $F^W_{N_W(D)}(\gamma_{p'})$ belongs to the Brauer correspondent $\widetilde{b}$ of b in $N_W(D)$. Next, since γ_p is p-special, then so is $(\gamma_p)_{N_W(D)}$ by [3, Proposition 6.1]. Also, as $\gamma_{p'}$ is p'-special, we know that $F^W_{N_W(D)}(\gamma_{p'})$ is p'-special. Then Lemma 4.4 tells us that the p-factorable character $(\gamma_p)_{N_W(D)} F^W_{N_W(D)}(\gamma_{p'})$ belongs to $\widetilde{b}$.

Now $\Omega_{G,D}(\chi) = ((\gamma_p)_{N_W(D)} F^W_{N_W(D)}(\gamma_{p'}))^{N_G(D)}$. Then by [10, Lemma 5.3.1], $\widetilde{b}^{N_G(D)}$ is defined and $\Omega_{G,D}(\chi) \in \mathrm{Irr}(\widetilde{b}^{N_G(D)})$. Since $\widetilde{b}^W = b$ and $b^G = B$, Lemma 5.3.4 of [10] says that $\widetilde{b}^G$ is defined and equals B. Then by the same lemma, we conclude that $(\widetilde{b}^{N_G(D)})^G$ is defined and equals B. It follows that $\widetilde{b}^{N_G(D)} = \widetilde{B}$, and therefore we have $\Omega_{G,D}(\chi) \in \mathrm{Irr}(\widetilde{B})$.

Next, since $\gamma_p(1) = 1$ and $D \in \mathrm{Syl}_p(N_W(D))$, we have

$$\big(\Omega_{G,D}(\chi)(1)\big)_p = \big|N_G(D) : N_W(D)\big|_p = \big|N_G(D) : D\big|_p.$$

This says that $\Omega_{G,D}(\chi) \in \mathrm{Irr}_0(\widetilde{B})$, and the proof of the theorem is complete. $\square$

5. Proof of the main theorems

In this section we prove Theorems A and B. Toward that goal, we need a number of preliminary results.

Let $N \lhd G$ where G has odd order, and let $Q \in \mathrm{Syl}_p(N)$. Then there is a Sylow p-subgroup P of G such that $P \cap N = Q$. Suppose M is a normal subgroup of G containing N. Then $P \cap M \in \mathrm{Syl}_p(M)$.

Now $(P\cap M)\cap N = Q$ and hence $N_M(P\cap M)\subseteq N_M(Q)$. Then the map $F^M_{N_M(Q)}$ is defined. Also, if α is a p-special character of M, the restriction $\alpha_{N_M(Q)}$ is a p-special character by [3, Proposition 6.1]. With these observations, we have:

Lemma 5.1. *Let $N\subseteq M$ be normal subgroups of a group G of odd order, and let $Q\in \mathrm{Syl}_p(N)$. Suppose μ is a G-invariant p-factorable character of N and let $\widetilde{\mu} = (\mu_p)_{N_N(Q)}F^N_{N_N(Q)}(\mu_{p'})$.*

(i) *If η is a p-factorable character of $N_G(Q)$ that lies over $\widetilde{\mu}$, then there exists a unique p-factorable character ζ of G such that $\eta = (\zeta_p)_{N_G(Q)}F^G_{N_G(Q)}(\zeta_{p'})$.*

(ii) *If θ is a p-factorable character of M, $\widetilde{\theta} = (\theta_p)_{N_M(Q)}F^M_{N_M(Q)}(\theta_{p'})$ and I is the inertial group of θ in G, then*:

(a) *$N_I(Q)$ $(= I\cap N_G(Q))$ is the inertial group of $\widetilde{\theta}$ in $N_G(Q)$.*

(b) *If χ is a p-factorable character of G, then χ lies over θ if and only if the p-factorable character $\widetilde{\chi} = (\chi_p)_{N_G(Q)}F^G_{N_G(Q)}(\chi_{p'})$ lies over $\widetilde{\theta}$.*

Proof. (i) By the Frattini argument $G = NN_G(Q)$. Next as μ is invariant in G, then [3, Proposition 7.1] implies that μ_p is invariant in G. Also $N\cap N_G(Q) = N_N(Q)$ and $(\mu_p)_{N_N(Q)}$ is irreducible. Then, in view of [7, Corollary 4.2], as $\eta_p\in \mathrm{Irr}(N_G(Q)|(\mu_p)_{N_N(Q)})$, η_p extends to G. Next by [8, Theorem F], η_p extends to a p-special character ϕ of G. Furthermore, by [3, Proposition 6.1], ϕ is the unique p-special extension of η_p to G.

Next by Lemma 2.1(ii), there exists a unique p'-special character ψ of G for which $\eta_{p'} = F^G_{N_G(Q)}(\psi)$. Let $\zeta = \phi\psi$. Then [3, Proposition 7.1] implies that ζ is the unique p-factorable character of G such that $\eta = (\zeta_p)_{N_G(Q)}F^G_{N_G(Q)}(\zeta_{p'})$.

(ii) Let $x\in N_I(Q)$. Then $(\theta_p)^x = (\theta^x)_p = \theta_p$, and hence $((\theta_p)_{N_M(Q)})^x = (\theta_p)_{N_M(Q)}$. Also, in view of Lemma 2.1(i),

$$F^M_{N_M(Q)}(\theta_{p'})^x = F^M_{N_M(Q)}\big((\theta_{p'})^x\big) = F^M_{N_M(Q)}\big(\big(\theta^x\big)_{p'}\big) = F^M_{N_M(Q)}(\theta_{p'}).$$

It follows that $\widetilde{\theta}^x = \widetilde{\theta}$. Next let $y\in \mathrm{I}_{N_G(Q)}(\widetilde{\theta})$, the stabilizer of $\widetilde{\theta}$ in $N_G(Q)$. Then $((\theta^y)_p)_{N_M(Q)} = (\theta_p)_{N_M(Q)}$, and hence $(\theta^y)_p = \theta_p$ by [3, Proposition 6.1]. Also, using Lemma 2.1(i) again, we have $F^M_{N_M(Q)}(\theta_{p'}) = F^M_{N_M(Q)}((\theta^y)_{p'})$, and hence by Lemma 2.1(ii), we get $(\theta^y)_{p'} = \theta_{p'}$. It follows that $\theta^y = \theta$. We have thus proved (a). Next, we prove (b).

Let P be a Sylow p-subgroup of G such that $P\cap N = Q$. If $R = P\cap M$, then $R\in \mathrm{Syl}_p(M)$. Lemma 3.5 of [14] says that $\chi_{p'}$ lies over $\theta_{p'}$ if and only if $F^G_{N_G(R)}(\chi_{p'})$ lies over $F^M_{N_M(R)}(\theta_{p'})$.

Next, we have $P\in \mathrm{Syl}_p(N_G(Q))$, $R\in \mathrm{Syl}_p(N_M(Q))$, $N_{N_G(Q)}(P) = N_G(P)$ and $N_{N_M(Q)}(R) = N_M(R)$. Now let $\beta = F^G_{N_G(Q)}(\chi_{p'})(\in \mathcal{X}_{p'}(N_G(Q)))$ and $\varepsilon = F^M_{N_M(Q)}(\theta_{p'})(\in \mathcal{X}_{p'}(N_M(Q)))$. Then $F^{N_G(Q)}_{N_G(P)}(\beta) = F^G_{N_G(P)}(\chi_{p'})$ and $F^{N_M(Q)}_{N_M(R)}(\varepsilon) = F^M_{N_M(R)}(\theta_{p'})$.

Next we have $N_M(Q)\lhd N_G(Q)$ and

$$N_{N_G(Q)}\big(P\cap N_M(Q)\big) = N_G(Q)\cap N_G\big(P\cap M\cap N_G(Q)\big) = N_G(Q)\cap N_G(R) = N_G(R).$$

Also $P \in \mathrm{Syl}_p(N_G(R))$, $N_{N_G(R)}(P) = N_G(P)$ and

$$\begin{aligned} F^{N_G(R)}_{N_G(P)}\big(F^{G}_{N_G(R)}(\chi_{p'})\big) &= F^{G}_{N_G(P)}(\chi_{p'}) \\ &= F^{N_G(Q)}_{N_G(P)}(\beta). \end{aligned}$$

Then by [14, Lemma 3.5] (with $N_G(Q)$, $N_M(Q)$, β, ε in place of G, N, χ and θ, respectively), we have that β lies over ε if and only if $F^{G}_{N_G(R)}(\chi_{p'})$ lies over $F^{N_M(Q)}_{N_M(R)}(\varepsilon)$ $(= F^{M}_{N_M(R)}(\theta_{p'}))$. It follows that $\chi_{p'}$ lies over $\theta_{p'}$ if and only if β lies over ε.

Now, to finish the proof, it suffices to show that χ_p lies over θ_p if and only if $(\chi_p)_{N_G(Q)}$ lies over $(\theta_p)_{N_M(Q)}$.

Suppose χ_p lies over θ_p. Since $((\chi_p)_M)_{N_M(Q)} = ((\chi_p)_{N_G(Q)})_{N_M(Q)}$, we have that $(\chi_p)_{N_G(Q)}$ lies over $(\theta_p)_{N_M(Q)}$. Next assume $(\chi_p)_{N_G(Q)}$ lies over $(\theta_p)_{N_M(Q)}$. As χ_p is p-special, we have $(\chi_p)_M = \xi_1 + \cdots + \xi_r$ for p-special characters $\xi_1, \ldots, \xi_r$ of M. Then $(\chi_p)_{N_M(Q)} = (\xi_1)_{N_M(Q)} + \cdots + (\xi_r)_{N_M(Q)}$. Now by [3, Proposition 6.1], every $(\xi_i)_{N_M(Q)}$ is p-special. Hence $(\theta_p)_{N_M(Q)} = (\xi_{i_0})_{N_M(Q)}$ for some i_0. Then [3, Proposition 6.1], again, implies that $\theta_p = \xi_{i_0}$. It follows that χ_p lies over θ_p. This clearly finishes the proof of (b). $\square$

Lemma 5.2. *Let G be a group of odd order and let $Q \subseteq N$ be subgroups of G such that $N \lhd G$ and Q is a p-group. Suppose $\mu \in \mathrm{Irr}(N|Q)$ and let $\widetilde{\mu} = \Omega_{N,Q}(\mu)$. If I is the inertial group of μ in G, then $N_I(Q)$ is the inertial group of $\widetilde{\mu}$ in $N_G(Q)$.*

Proof. Let $x \in N_G(Q)$. If $x \in I$, then in view of Lemma 4.2, $\widetilde{\mu}^x = \Omega_{N,Q}(\mu) = \widetilde{\mu}$. On the other hand, suppose $\widetilde{\mu}^x = \widetilde{\mu}$. Then again by Lemma 4.2, $\mu^x \in \mathrm{Irr}(N|Q)$ and $\Omega_{N,Q}(\mu^x) = \widetilde{\mu}$. Theorem 4.1 now implies that $\mu^x = \mu$. The result is then immediate. $\square$

Lemma 5.3. *Let G be a group of odd order and let $Q \subseteq N$ be subgroups of G where $N \lhd G$ and Q is a p-group. Suppose $\mu \in \mathrm{Irr}(N|Q)$ is G-invariant and let $\widetilde{\mu} = \Omega_{N,Q}(\mu)$. Let (W, γ) be a normal nucleus for μ such that $Q \in \mathrm{Syl}_p(W)$, and let $\widetilde{\gamma} = (\gamma_p)_{N_W(Q)} F^{W}_{N_W(Q)}(\gamma_{p'})$ (so that $\widetilde{\mu} = \widetilde{\gamma}^{N_N(Q)}$). If S and $\widetilde{S}$ are the respective stabilizers of (W, γ) in G and $(N_W(Q), \widetilde{\gamma})$ in $N_G(Q)$, then*:

(i) *$G = SN$, $W = S \cap N$ and character induction defines a bijection of* $\mathrm{Irr}(S|\gamma)$ *onto* $\mathrm{Irr}(G|\mu)$.
(ii) *$N_G(Q) = \widetilde{S} N_N(Q)$, $N_W(Q) = \widetilde{S} \cap N_N(Q)$ and $\widetilde{S} = N_S(Q)$. Moreover, character induction defines a bijection of* $\mathrm{Irr}(\widetilde{S}|\widetilde{\gamma})$ *onto* $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$.

Proof. (i) Let $g \in G$. As μ is invariant in G, $(W, \gamma)^g$ is a normal nucleus of μ. But any two normal nuclei of μ are N-conjugate. Therefore $(W, \gamma)^g = (W, \gamma)^n$ for some $n \in N$, and hence $gn^{-1} \in S$. It then follows that $G = SN$.

Next as $\mu = \gamma^N$ and W is normal in $S \cap N$, we have $\nu = \gamma^{S \cap N} \in \mathrm{Irr}(S \cap N)$ and $\nu_W = \nu(1)\gamma(1)^{-1}\gamma$. Also

$$\nu(1)\gamma(1)^{-1} = [\nu_W, \gamma] = \big[\nu, \gamma^{S \cap N}\big] = 1.$$

It follows that $\nu(1) = \gamma(1)$ and therefore $W = S \cap N$.

The last statement follows from Corollary 4.3 in [7].

(ii) Let $h \in N_G(Q)$. Then as in (i), $(W, \gamma)^h = (W, \gamma)^m$ for some $m \in N$. Now $W = W^{hm^{-1}}$. As $Q \in \mathrm{Syl}_p(W)$, we have $Q^{m^{-1}} = Q^{hm^{-1}} \in \mathrm{Syl}_p(W)$, also. Then, by Sylow theory, $Q^{m^{-1}} = Q^w$ for some $w \in W$, and hence $wm \in N_G(Q)$. But $w \in W \subseteq N$ and $m \in N$. It follows that $wm \in N_N(Q)$. So, if $l = wm$, we now have $W^h = W^l$ with $l \in N_N(Q)$. Then

$$N_W(Q)^h = N_{W^h}(Q) = N_{W^l}(Q) = N_W(Q)^l.$$

Also,

$$\begin{aligned}\widetilde{\gamma}^h &= \big((\gamma^h)_p\big)_{N_{W^h}(Q)} F^{W^h}_{N_{W^h}(Q)}\big((\gamma^h)_{p'}\big)\\ &= \big((\gamma^l)_p\big)_{N_{W^l}(Q)} F^{W^l}_{N_{W^l}(Q)}\big((\gamma^l)_{p'}\big)\\ &= \widetilde{\gamma}^l.\end{aligned}$$

Therefore $hl^{-1} \in \widetilde{S}$. This shows that $N_G(Q) = \widetilde{S}N_N(Q)$. Next, as $\widetilde{\mu} = \widetilde{\gamma}^{N_N(Q)}$, an argument similar to that used in (i) to show that $W = S \cap N$ (with $\widetilde{\gamma}$ in place of γ), gives $N_W(Q) = \widetilde{S} \cap N_N(Q)$.

Next we show that $\widetilde{S} = N_S(Q)$. Suppose $k \in N_S(Q)$. Then $(W, \gamma)^k = (W, \gamma)$, and hence we easily see that $(N_W(Q), \widetilde{\gamma})^k = (N_W(Q), \widetilde{\gamma})$. Therefore $N_S(Q) \subseteq \widetilde{S}$.

As $G = SN$ and $W = S \cap N$ (by (i)), we have $|G|/|N| = |S|/|W|$. Similarly, since $N_G(Q) = \widetilde{S}N_N(Q)$ and $N_W(Q) = \widetilde{S} \cap N_N(Q)$, we get $|N_G(Q)|/|N_N(Q)| = |\widetilde{S}|/|N_W(Q)|$. Next, we claim that $G = N_G(Q)N$.

Let $g \in G$. As $\mu \in \mathrm{Irr}(N|Q)$ and μ is invariant in G, we have $\mu \in \mathrm{Irr}(N|Q^g)$. But the vertices of μ are N-conjugate. Therefore $Q^g = Q^n$ for some $n \in N$, and hence $gn^{-1} \in N_G(Q)$. This shows that $G = N_G(Q)N$, and our claim is valid.

We now have $|G|/|N| = |N_G(Q)|/|N_N(Q)|$, and then $|S|/|W| = |\widetilde{S}|/|N_W(Q)|$. Next, as $Q \in \mathrm{Syl}_p(W)$ and S normalizes W, we get $S = N_S(Q)W$ by the Frattini argument. Then $|S|/|W| = |N_S(Q)|/|N_W(Q)|$. It follows that $|\widetilde{S}| = |N_S(Q)|$. Now since $N_S(Q) \subseteq \widetilde{S}$, we conclude that $\widetilde{S} = N_S(Q)$.

The last assertion is immediate from [7, Corollary 4.3]. □

Our next proposition is an important step toward the construction of the map of Theorem A. We should mention that in some parts of the proof of this proposition, we adapt some arguments of Navarro, which he used to construct the character injection in Theorem 5.4 of [14].

Proposition 5.4. *Let $N \lhd G$ where G is a group of odd order and let $Q \in \mathrm{Syl}_p(N)$. Let μ be a G-invariant p-factorable character of N and write $\widetilde{\mu}$ for the p-factorable character $(\mu_p)_{N_N(Q)} F^N_{N_N(Q)}(\mu_{p'})$. Then there is a natural bijection of* $\mathrm{Irr}(G|\mu)$ *onto* $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$.

Proof. First note that for any subgroup H of G containing N, $H = NN_H(Q)$ by the Frattini argument.

Let $\chi \in \mathrm{Irr}(G|\mu)$. Next let (W, γ) be a normal nucleus for χ. Since μ is G-invariant and p-factorable, it follows by the construction of the normal nucleus that $N \subseteq W$ and γ lies over μ.

Let $R \in \mathrm{Syl}_p(W)$ with $Q \subseteq R$. Then $Q = R \cap N$, and hence $N_W(R) \subseteq N_W(Q)$. By [3, Proposition 6.1], $(\gamma_p)_{N_W(Q)}$ is a p-special character of $N_W(Q)$. Also, the map $F^W_{N_W(Q)}$ is defined, and by Lemma 2.1(ii), $F^W_{N_W(Q)}(\gamma_{p'})$ is a p'-special character of $N_W(Q)$. Let $\widetilde{\gamma} = (\gamma_p)_{N_W(Q)} F^W_{N_W(Q)}(\gamma_{p'})$. Then $\widetilde{\gamma}$ is a p-factorable character of $N_W(Q)$. Now let $\widetilde{\chi} = \widetilde{\gamma}^{N_G(Q)}$. The objective now is to show that the correspondence $\chi \mapsto \widetilde{\chi}$ defines the desired natural bijection from $\mathrm{Irr}(G|\mu)$ onto $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$.

Step 1. $\widetilde{\chi}$ is independent of the choice of the normal nucleus (W, γ).

Let (W', γ') be a normal nucleus for χ. Then $(W', \gamma') = (W, \gamma)^g$ for some $g \in G$. Since $G = NN_G(Q)$, we may assume that $g \in N_G(Q)$. Now

$$\big(\gamma'_p\big)_{N_{W'}(Q)} = \big((\gamma_p)^g\big)_{N_{W^g}(Q)} = \big((\gamma_p)_{N_W(Q)}\big)^g.$$

Also, by Lemma 2.1(i),

$$F^{W'}_{N_{W'}(Q)}\big(\gamma'_{p'}\big) = F^{W^g}_{N_{W^g}(Q)}\big((\gamma_{p'})^g\big) = F^W_{N_W(Q)}(\gamma_{p'})^g.$$

Then $((\gamma'_p)_{N_{W'}(Q)} F^{W'}_{N_{W'}(Q)}(\gamma'_{p'}))^{N_G(Q)} = \widetilde{\gamma}^{N_G(Q)}$. This shows that $\widetilde{\chi}$ only depends on χ.

Step 2. $\widetilde{\chi}$ lies in $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$ and has normal nucleus $(N_W(Q),\widetilde{\gamma})$.

We prove this by induction on $|G|$. If χ is p-factorable, then $(G,\chi)=(W,\gamma)$. In this case $\widetilde{\chi}=\widetilde{\gamma}$ is p-factorable, hence irreducible, and $(N_W(Q),\widetilde{\gamma})$ is clearly the sole nucleus for $\widetilde{\chi}$. Next, by Lemma 5.1(ii) (with W, N, γ in place of G, M and χ, respectively), as γ lies over μ, we have $\widetilde{\chi}$ $(=\widetilde{\gamma})$ lies over $\widetilde{\mu}$.

Now suppose that χ is not p-factorable. By the construction of the normal nucleus, there exists a maximal p-factorable normal pair (M,θ) of G with $M\subseteq W$, $\gamma\in\mathrm{Irr}(W|\theta)$, and such that if I is the inertial group of θ in G, then $W\subseteq I<G$, $\psi=\gamma^I\in\mathrm{Irr}(I|\theta)$ and ψ has normal nucleus (W,γ).

As (M,θ) is a maximal p-factorable normal pair under χ and μ is p-factorable and G-invariant, we have that $N\subseteq M$ and $\theta\in\mathrm{Irr}(M|\mu)$. It follows, in particular, that $\psi\in\mathrm{Irr}(I|\mu)$. Since $I<G$, the inductive hypothesis guarantees that $\widetilde{\psi}=\widetilde{\gamma}^{N_I(Q)}\in\mathrm{Irr}(N_I(Q)|\widetilde{\mu})$ and $(N_W(Q),\widetilde{\gamma})$ is a normal nucleus of $\widetilde{\psi}$.

By Lemma 5.1(ii), $N_I(Q)$ is the stabilizer of the p-factorable character $\widetilde{\theta}=(\theta_p)_{N_M(Q)}F^{M}_{N_M(Q)}(\theta_{p'})$ in $N_G(Q)$. Next, since γ lies over θ, then again by Lemma 5.1(ii) (with W in place of G), $\widetilde{\gamma}$ lies over $\widetilde{\theta}$, and hence $\widetilde{\psi}\in\mathrm{Irr}(N_I(Q)|\widetilde{\theta})$. Now by the Clifford correspondence (see [6, Theorem 6.11]), $\widetilde{\chi}=\widetilde{\psi}^{N_G(Q)}\in\mathrm{Irr}(N_G(Q)|\widetilde{\theta})$. Furthermore, as $\widetilde{\psi}$ lies over $\widetilde{\mu}$, then so does $\widetilde{\chi}$. Next, we show that $(N_W(Q),\widetilde{\gamma})$ is a normal nucleus for $\widetilde{\chi}$.

We claim that $(N_M(Q),\widetilde{\theta})$ is a maximal p-factorable normal pair of $N_G(Q)$. Suppose, by way of contradiction, that there exists a p-factorable normal pair (K,ω) in $N_G(Q)$ above $(N_M(Q),\widetilde{\theta})$ with $K\neq N_M(Q)$.

Let $J=NK$. Then, as $K\lhd N_G(Q)$ and $G=NN_G(Q)$, we have $J\lhd G$. Next

$$N_J(Q)=(NK)\cap N_G(Q)=KN_N(Q)=K,$$

and so $J=NN_J(Q)$. Now $|J|=(|N||K|)/|N_N(Q)|$, and as $M=NN_M(Q)$, $|M|=(|N||N_M(Q)|)/|N_N(Q)|$. Since $N_M(Q)<K$, it follows that $M<J$.

Since θ lies over μ, part (b) of Lemma 5.1(ii) tells us that $\widetilde{\theta}$ lies over $\widetilde{\mu}$. Then ω lies over $\widetilde{\mu}$, as ω lies over $\widetilde{\theta}$. Now, by Lemma 5.1(i), there exists a p-factorable character ζ of J for which $\omega=(\zeta_p)_{N_J(Q)}F^{J}_{N_J(Q)}(\zeta_{p'})$. Next part (b) of Lemma 5.1(ii) says that ζ lies over θ. But now (J,ζ) is a p-factorable normal pair of G lying strictly above (M,θ). Therefore $(N_M(Q),\widetilde{\theta})$ must be a maximal p-factorable normal pair of $N_G(Q)$, as claimed.

Next, as χ is not p-factorable, we have $M<G$. Now since $M=NN_M(Q)$ and $G=NN_G(Q)$, it follows that $N_M(Q)<N_G(Q)$. So, in particular, $\widetilde{\chi}$ is not p-factorable.

Let us now recapitulate. The pair $(N_M(Q),\widetilde{\theta})$ is a maximal p-factorable normal pair under $\widetilde{\chi}$, the subgroup $N_I(Q)$ is the inertial group of $\widetilde{\theta}$ in $N_G(Q)$, and $\widetilde{\psi}(\in\mathrm{Irr}(N_I(Q)|\widetilde{\theta}))$ is the Clifford correspondent of $\widetilde{\chi}$. Since $(N_W(Q),\widetilde{\gamma})$ is a normal nucleus of $\widetilde{\psi}$, it follows (by the construction of the normal nucleus) that $(N_W(Q),\widetilde{\gamma})$ is a normal nucleus for $\widetilde{\chi}$.

Step 3. The map $\chi\mapsto\widetilde{\chi}$ is one-to-one.

Let $\chi_0\in\mathrm{Irr}(G|\mu)$ and suppose $\widetilde{\chi_0}=\widetilde{\chi}$. Next choose a normal nucleus (W_0,γ_0) for χ_0, and note that W_0 contains N (see the beginning of the proof).

Let $\widetilde{\gamma_0}=((\gamma_0)_p)_{N_{W_0}(Q)}F^{W_0}_{N_{W_0}(Q)}((\gamma_0)_{p'})$. Then, by Step 2, $(N_W(Q),\widetilde{\gamma})$ and $(N_{W_0}(Q),\widetilde{\gamma_0})$ are both nuclei for $\widetilde{\chi}$. Hence $(N_{W_0}(Q),\widetilde{\gamma_0})=(N_W(Q),\widetilde{\gamma})^x$ for some $x\in N_G(Q)$. Since $W=NN_W(Q)$ and $W_0=NN_{W_0}(Q)$, we deduce that $W_0=W^x$. Next, as $\widetilde{\gamma_0}=\widetilde{\gamma}^x$, we get $((\gamma_0)_p)_{N_{W^x}(Q)}=((\gamma^x)_p)_{N_{W^x}(Q)}$ and $F^{W^x}_{N_{W^x}(Q)}((\gamma_0)_{p'})=F^{W^x}_{N_{W^x}(Q)}((\gamma^x)_{p'})$. Then $(\gamma_0)_p=(\gamma^x)_p$ by [3, Proposition 6.1] and $(\gamma_0)_{p'}=(\gamma^x)_{p'}$ by Lemma 2.1(ii). It follows that $\gamma_0=\gamma^x$. Now

$$\chi_0=\gamma_0^G=\left(\gamma^x\right)^G=\gamma^G=\chi.$$

This shows that the map $\chi\mapsto\widetilde{\chi}$ is one-to-one.

Step 4. The map $\chi \mapsto \widetilde{\chi}$ is onto.

First, in view of part (a) of Lemma 5.1(ii), note that $\widetilde{\mu}$ is invariant in $N_G(Q)$. Next let $\xi \in \mathrm{Irr}(N_G(Q)|\widetilde{\mu})$. Now choose a normal nucleus (V,τ) for ξ. Since $\widetilde{\mu}$ is $N_G(Q)$-invariant and p-factorable, we have $N_N(Q) \subseteq V$ and τ lies over $\widetilde{\mu}$.

Now let $V' = NV$. Then

$$N_{V'}(Q) = (NV) \cap N_G(Q) = V N_N(Q) = V.$$

By Lemma 5.1(i), there exists a unique p-factorable character τ' of V' such that $\tau = (\tau'_p)_V F_V^{V'}(\tau'_{p'})$.

Let $\xi' = (\tau')^G$. We prove by induction on $|G|$ that $\xi' \in \mathrm{Irr}(G|\mu)$ and that (V',τ') is a normal nucleus for ξ'.

Suppose ξ is p-factorable. Then $(N_G(Q),\xi) = (V,\tau)$, and so, in particular,

$$G = NN_G(Q) = NV = V'.$$

Now clearly $\xi' = \tau'$ is irreducible and (V',τ') is the (sole) normal nucleus of ξ'. Next, as τ lies over $\widetilde{\mu}$, part (b) of Lemma 5.1(ii) tells us that ξ' $(=\tau')$ lies over μ.

Suppose now that ξ is not p-factorable. By the construction of the normal nucleus, we can find a maximal p-factorable normal pair (L,η) of $N_G(Q)$ with $L \subseteq V$, $\tau \in \mathrm{Irr}(V|\eta)$ and such that if T is the inertial group of η in $N_G(Q)$, then $V \subseteq T < N_G(Q)$, $\phi = \tau^T \in \mathrm{Irr}(T|\eta)$ and ϕ has normal nucleus (V,τ).

Let $T' = NT$. As $N_N(Q) \subseteq V \subseteq T$, we have

$$N_{T'}(Q) = N_G(Q) \cap T' = T\big(N \cap N_G(Q)\big) = TN_N(Q) = T.$$

Then $|T'| = (|N||T|)/|N_N(Q)|$. Since $|G| = (|N||N_G(Q)|)/|N_N(Q)|$ and $T < N_G(Q)$, we conclude that $|T'| < |G|$.

Since (L,η) is a maximal p-factorable normal pair of $N_G(Q)$ under ξ and $\widetilde{\mu}$ is p-factorable and $N_G(Q)$-invariant, we have that $N_N(Q) \subseteq L$ and η lies over $\widetilde{\mu}$. Therefore $\phi \in \mathrm{Irr}(N_{T'}(Q)|\widetilde{\mu})$.

Now, by induction, we have $\phi' = (\tau')^{T'} \in \mathrm{Irr}(T'|\mu)$ and (V',τ') is a normal nucleus for ϕ'.

Let $L' = NL$. Then one can easily see that $L = N_{L'}(Q)$. Since η lies over $\widetilde{\mu}$, then, in light of Lemma 5.1(i), there exists a unique p-factorable character η' of L' for which $\eta = (\eta'_p)_{N_{L'}(Q)} F^{L'}_{N_{L'}(Q)}(\eta'_{p'})$.

Since $L \lhd N_G(Q)$ and $G = NN_G(Q)$, note that $L' \lhd G$. We claim that T' is the stabilizer of η' in G.

Suppose S is the stabilizer of η' in G. Then, by part (a) of Lemma 5.1(ii), $N_S(Q)$ is the inertial group of η in $N_G(Q)$. Therefore $N_S(Q) = T$. Now we have

$$S = S \cap NN_G(Q) = NT = T',$$

and our claim is valid.

Next, $L' = NL \subseteq NV = V'$, and τ lies over η. Then τ' lies over η' by part (b) of Lemma 5.1(ii). Since $\phi' = (\tau')^{T'}$, we get that $\phi' \in \mathrm{Irr}(T'|\eta')$.

Now by the Clifford correspondence, $\xi' = (\phi')^G \in \mathrm{Irr}(G|\eta')$. Also, as ϕ' lies over μ, then so does ξ'. Next we show that (V',τ') is a normal nucleus of ξ'.

We claim that (L',η') is a maximal p-factorable normal pair in G. Assume, on the contrary, that there exists a p-factorable normal pair (E,λ) of G such that $(L',\eta') \leqslant (E,\lambda)$ and $L' \neq E$. Let $\widetilde{\lambda}$ be the p-factorable character $(\lambda_p)_{N_E(Q)} F^E_{N_E(Q)}(\lambda_{p'})$ of $N_E(Q)$.

Since λ lies over η', part (b) of Lemma 5.1(ii) says that $\widetilde{\lambda}$ lies over η. Next as $L' = NN_{L'}(Q)$ and $E = NN_E(Q)$, we have $|L'| = (|N||N_{L'}(Q)|)/|N_N(Q)|$ and $|E| = (|N||N_E(Q)|)/|N_N(Q)|$. Since $L' < E$, it follows that $L = N_{L'}(Q) < N_E(Q)$. Now $(N_E(Q),\widetilde{\lambda})$ is a p-factorable normal pair of $N_G(Q)$ lying strictly above (L,η), and this clearly contradicts the maximality of (L,η). Therefore (L',η') must be a maximal p-factorable normal pair of G, as claimed.

Now by the construction of the normal nucleus, since (V',τ') is a normal nucleus of ϕ', then it is one for ξ'.

Finally, as ξ' lies over μ and has (V',τ') as a normal nucleus, we have

$$\begin{aligned}\widetilde{\xi'} &= \left((\tau'_p)_{N_{V'}(Q)} F^{V'}_{N_{V'}(Q)}(\tau'_{p'})\right)^{N_G(Q)}\\ &= \tau^{N_G(Q)}\\ &= \xi.\end{aligned}$$

This shows that the map $\chi \mapsto \widetilde{\chi}$ is onto. The proof of the proposition is now complete. $\square$

The following result includes Theorem A.

Theorem 5.5. *Let $N \lhd G$ where G is a group of odd order and let $\mu \in \mathrm{Irr}(N|Q)$, where Q is a p-subgroup of N. Let $\widetilde{\mu} = \Omega_{N,Q}(\mu)$.*

(i) *There is a natural p-defect-preserving bijection Δ from $\mathrm{Irr}(G|\mu)$ onto $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$.*
(ii) *If $\chi \in \mathrm{Irr}(G|\mu)$ belongs to the p-block B of G, then the character $\Delta(\chi)$ belongs to a p-block $\widetilde{B}$ of $N_G(Q)$ such that $\widetilde{B}^G = B$.*

Proof. Let T be the inertial group of μ in G. We claim that $N_T(Q)$ is the inertial group of $\widetilde{\mu}$ in $N_G(Q)$. Suppose $x \in N_T(Q)$. Then $\mu^x = \mu$, and by Lemma 4.2, it follows that $\widetilde{\mu}^x = \widetilde{\mu}$. On the other hand, assume $\widetilde{\mu}^y = \widetilde{\mu}$ with $y \in N_G(Q)$. Then, in view of Lemma 4.2, $\widetilde{\mu} = \Omega_{N,Q}(\mu^y)$. Since the map $\Omega_{N,Q}$ is injective (by Theorem 4.1), we must have $\mu^y = \mu$. This shows that the stabilizer of $\widetilde{\mu}$ in $N_G(Q)$ is $N_T(Q)$, as claimed.

In view of [6, Theorem 6.11], character induction defines bijections Υ from $\mathrm{Irr}(T|\mu)$ onto $\mathrm{Irr}(G|\mu)$ and $\widetilde{\Upsilon}$ from $\mathrm{Irr}(N_T(Q)|\widetilde{\mu})$ onto $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$.

Let (W,γ) be a normal nucleus for μ such that $Q \in \mathrm{Syl}_p(W)$, and let $\widetilde{\gamma} = (\gamma_p)_{N_W(Q)} F^W_{N_W(Q)}(\gamma_{p'})$ (so that $\widetilde{\mu} = \widetilde{\gamma}^{N_N(Q)}$). Now let S be the stabilizer of (W,γ) in T. Then, by Lemma 5.3(ii), $N_S(Q)$ is the stabilizer of $(N_W(Q),\widetilde{\gamma})$ in $N_T(Q)$.

By Lemma 5.3, character induction defines bijections $\Theta_{W,\gamma}$ from $\mathrm{Irr}(S|\gamma)$ onto $\mathrm{Irr}(T|\mu)$ and $\widetilde{\Theta}_{W,\gamma}$ from $\mathrm{Irr}(N_S(Q)|\widetilde{\gamma})$ onto $\mathrm{Irr}(N_T(Q)|\widetilde{\mu})$. Next, by Proposition 5.4, there exists a natural bijection $\Phi_{W,\gamma}$ from $\mathrm{Irr}(S|\gamma)$ onto $\mathrm{Irr}(N_S(Q)|\widetilde{\gamma})$.

Our first task is to show that the bijection $\widetilde{\Theta}_{W,\gamma} \circ \Phi_{W,\gamma} \circ (\Theta_{W,\gamma})^{-1}$ does not depend on the choice of the normal nucleus (W,γ).

Let $\psi \in \mathrm{Irr}(T|\mu)$. Next let ξ be the character in $\mathrm{Irr}(S|\gamma)$ such that $\Theta_{W,\gamma}(\xi) = \psi$. Now choose a normal nucleus (U,ε) for ξ. Then, by the proof of Proposition 5.4, we have $W \subseteq U$, $\widetilde{\varepsilon} = (\varepsilon_p)_{N_U(Q)} F^U_{N_U(Q)}(\varepsilon_{p'})$ is a p-factorable character of $N_U(Q)$ and $\Phi_{W,\gamma}(\xi) = \widetilde{\varepsilon}^{N_S(Q)}$. Now $(\widetilde{\Theta}_{W,\gamma} \circ \Phi_{W,\gamma} \circ (\Theta_{W,\gamma})^{-1})(\psi) = \widetilde{\varepsilon}^{N_T(Q)}$.

Suppose now that (W',γ') is another normal nucleus of μ such that $Q \in \mathrm{Syl}_p(W')$. Then $(W',\gamma') = (W,\gamma)^n$ for some $n \in N$. Since Q and $Q^{n^{-1}}$ are both Sylow p-subgroups of W, then (by Sylow theory) $Q^{n^{-1}} = Q^w$ for some $w \in W$. It follows that $wn \in N_N(Q)$, and hence we may assume that $n \in N_N(Q)$.

It is easy to see that S^n is the stabilizer of (W',γ') in T. Then if $\widetilde{\gamma'} = (\gamma'_p)_{N_{W'}(Q)} F^{W'}_{N_{W'}(Q)}(\gamma'_{p'})$, we have $\widetilde{\gamma'} = \widetilde{\gamma}^n$, and the subgroup $N_{S^n}(Q)$ is the stabilizer of $(N_{W'}(Q),\widetilde{\gamma'})$ in $N_T(Q)$.

Next we have $\xi^n \in \mathrm{Irr}(S^n|\gamma')$ and $(\xi^n)^T = \psi$. Thus $\xi^n = (\Theta_{W',\gamma'})^{-1}(\psi)$. Since (U,ε) is a normal nucleus of ξ, then (by the construction of the normal nucleus), $(U,\varepsilon)^n$ is a normal nucleus for ξ^n. Now if $\widetilde{\varepsilon^n} = ((\varepsilon^n)_p)_{N_{U^n}(Q)} F^{U^n}_{N_{U^n}(Q)}((\varepsilon^n)_{p'})$, we have $(\widetilde{\Theta}_{W',\gamma'} \circ \Phi_{W',\gamma'} \circ (\Theta_{W',\gamma'})^{-1})(\psi) = (\widetilde{\varepsilon^n})^{N_T(Q)}$. But $((\varepsilon^n)_p)_{N_{U^n}(Q)} = ((\varepsilon_p)_{N_U(Q)})^n$ and by Lemma 2.1(i), we have $F^{U^n}_{N_{U^n}(Q)}((\varepsilon^n)_{p'}) = (F^U_{N_U(Q)}(\varepsilon_{p'}))^n$. Hence $\widetilde{\varepsilon^n} = \widetilde{\varepsilon}^n$ and it follows that $(\widetilde{\varepsilon^n})^{N_T(Q)} = \widetilde{\varepsilon}^{N_T(Q)}$. This shows that the bijection

$$\widetilde{\Theta}_{W,\gamma} \circ \Phi_{W,\gamma} \circ (\Theta_{W,\gamma})^{-1}$$

is independent of the choice of the normal nucleus (W,γ).

Now let

$$\Delta = \widetilde{\Upsilon} \circ \widetilde{\Theta}_{W,\gamma} \circ \Phi_{W,\gamma} \circ (\Theta_{W,\gamma})^{-1} \circ \Upsilon^{-1}.$$

Then Δ is a natural bijection of $\mathrm{Irr}(G|\mu)$ onto $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$. Next, as Υ, $\widetilde{\Upsilon}$, $\Theta_{W,\gamma}$ and $\widetilde{\Theta}_{W,\gamma}$ are all character induction maps, they preserve character defects. So, to prove that Δ preserves defects, it suffices to show that the map $\Phi_{W,\gamma}$ preserves defects.

Let $\zeta \in \mathrm{Irr}(S|\gamma)$. Then choose a normal nucleus (Y,η) for ζ. By the proof of Proposition 5.4, we have $W \subseteq Y$, $\widetilde{\eta} = (\eta_p)_{N_Y(Q)} F^{Y}_{N_Y(Q)}(\eta_{p'})$ is a p-factorable character of $N_Y(Q)$ and $\Phi_{W,\gamma}(\zeta) = \widetilde{\eta}^{N_S(Q)}$.

Since $N_Y(Q)$ contains a Sylow p-subgroup of Y, we have $|N_Y(Q)|_p = |Y|_p$. Next note that $\widetilde{\eta}(1)_p = \eta(1)_p$. Now as $\zeta(1) = |S:Y|\eta(1)$ and

$$\Phi_{W,\gamma}(\zeta)(1) = \bigl|N_S(Q):N_Y(Q)\bigr|\widetilde{\eta}(1),$$

it follows that $|S|_p/\zeta(1)_p = |N_S(Q)|_p/(\Phi_{W,\gamma}(\zeta)(1))_p$. This shows that $\Phi_{W,\gamma}$ preserves character defects. The proof of (i) is now complete. Next we take care of (ii).

Let $\chi \in \mathrm{Irr}(G|\mu)$. Next set $\rho = ((\Theta_{W,\gamma})^{-1} \circ \Upsilon^{-1})(\chi)$ and let B be the block of G to which χ belongs. Now choose a normal nucleus (X,σ) for ρ. Then $W \subseteq X$ and $\Phi_{W,\gamma}(\rho) = \widetilde{\sigma}^{N_S(Q)}$ where $\widetilde{\sigma}$ is the p-factorable character $(\sigma_p)_{N_X(Q)} F^{X}_{N_X(Q)}(\sigma_{p'})$ of $N_X(Q)$.

Let b (resp. $\widetilde{b}$) be the block of X (resp. $N_X(Q)$) to which σ (resp. $\widetilde{\sigma}$) belongs. Then, in view of Lemma 4.4, $\sigma_{p'}$ and $F^{X}_{N_X(Q)}(\sigma_{p'})$ belong to b and $\widetilde{b}$, respectively. Now Proposition 2.2 says that $\widetilde{b}^X = b$. Also, by [10, Lemma 5.3.1], as $\chi = \sigma^G$, we have that b^G is defined and equals B.

Next let $\widetilde{\chi} = \Delta(\chi)$ and let $\widetilde{B}$ be the block of $N_G(Q)$ to which $\widetilde{\chi}$ belongs. Then $\widetilde{\chi} = \widetilde{\sigma}^{N_G(Q)}$, and so again by [10, Lemma 5.3.1], we have that $\widetilde{b}^{N_G(Q)}$ is defined and equals $\widetilde{B}$.

Now by Lemma 5.3.4 of [10], $\widetilde{b}^G$ is defined and equals B. Then, using the same lemma once more, we conclude that $\widetilde{B}^G$ is defined and equals B. This clearly ends the proof of (ii). $\square$

Let $N \lhd G$ where G is a group of odd order. Suppose $\mu \in \mathrm{Irr}(N|Q)$ where Q is a p-subgroup of N, and let $\widetilde{\mu} = \Omega_{N,Q}(\mu)$. If $x \in N_G(Q)$, then by Lemma 4.2, $\mu^x \in \mathrm{Irr}(N|Q)$ and $\widetilde{\mu}^x = \Omega_{N,Q}(\mu^x)$. According to Theorem 5.5(i), we have natural bijections Δ_μ from $\mathrm{Irr}(G|\mu)$ onto $\mathrm{Irr}(N_G(Q)|\widetilde{\mu})$ and Δ_{μ^x} from $\mathrm{Irr}(G|\mu^x)$ onto $\mathrm{Irr}(N_G(Q)|\widetilde{\mu}^x)$. It is clear that $\mathrm{Irr}(G|\mu^x) = \mathrm{Irr}(G|\mu)$ and $\mathrm{Irr}(N_G(Q)|\widetilde{\mu}^x) = \mathrm{Irr}(N_G(Q)|\widetilde{\mu})$. The following shows that, in fact, Δ_μ and Δ_{μ^x} are the same map.

Lemma 5.6. *With the above assumptions and notation, we have $\Delta_{\mu^x} = \Delta_\mu$.*

Proof. Let $\chi \in \mathrm{Irr}(G|\mu)$. Write T for the inertial group of μ in G and let ψ be the unique character in $\mathrm{Irr}(T|\mu)$ such that $\chi = \psi^G$. Next choose a normal nucleus (W,γ) for μ such that $Q \in \mathrm{Syl}_p(W)$, and let S be its stabilizer in T. In view of Lemma 5.3(i), there exists a unique character ξ in $\mathrm{Irr}(S|\gamma)$ for which $\psi = \xi^T$. Next choose a normal nucleus (U,ε) for ξ, and let $\widetilde{\varepsilon} = (\varepsilon_p)_{N_U(Q)} F^{U}_{N_U(Q)}(\varepsilon_{p'})$. Then $\Delta_\mu(\chi) = \widetilde{\varepsilon}^{N_G(Q)}$.

Now let $x \in N_G(Q)$. It is clear that T^x is the inertial group of μ^x in G. Furthermore, ψ^x is the unique character in $\mathrm{Irr}(T^x|\mu^x)$ for which $\chi = (\psi^x)^G$. Next (W^x,γ^x) is a normal nucleus for μ^x and $Q \in \mathrm{Syl}_p(W^x)$ (as $x \in N_G(Q)$). Also, note that S^x is the stabilizer of (W^x,γ^x) in T^x. Since $\xi \in \mathrm{Irr}(S|\gamma)$ and $\xi^T = \psi$, we have $\xi^x \in \mathrm{Irr}(S^x|\gamma^x)$ and $(\xi^x)^{T^x} = \psi^x$. Next, as ξ has normal nucleus (U,ε), then ξ^x has normal nucleus (U^x,ε^x). Now if $\widetilde{\varepsilon^x} = ((\varepsilon^x)_p)_{N_{U^x}(Q)} F^{U^x}_{N_{U^x}(Q)}((\varepsilon^x)_{p'})$, then $\Delta_{\mu^x}(\chi) = (\widetilde{\varepsilon^x})^{N_G(Q)}$.

Since $(\varepsilon^x)_p = (\varepsilon_p)^x$, we have $((\varepsilon^x)_p)_{N_{U^x}(Q)} = ((\varepsilon_p)_{N_U(Q)})^x$. Also, as $(\varepsilon^x)_{p'} = (\varepsilon_{p'})^x$, we get $F^{U^x}_{N_{U^x}(Q)}((\varepsilon^x)_{p'}) = F^{U}_{N_U(Q)}(\varepsilon_{p'})^x$ by Lemma 2.1(i). It follows that $\widetilde{\varepsilon^x} = \widetilde{\varepsilon}^x$, and therefore $\Delta_{\mu^x}(\chi) = (\widetilde{\varepsilon}^x)^{N_G(Q)} = \widetilde{\varepsilon}^{N_G(Q)} = \Delta_\mu(\chi)$. This shows that $\Delta_{\mu^x} = \Delta_\mu$, as needed. $\square$

Let G be an arbitrary finite group and let N be a normal subgroup of G. Suppose B and b are p-blocks of G and N respectively such that B covers b. If $\mu \in \operatorname{Irr}(b)$, we write $\operatorname{Irr}(B|\mu)$ for the set $\operatorname{Irr}(B) \cap \operatorname{Irr}(G|\mu)$. We are finally ready to prove Theorem B.

Proof of Theorem B. Let $\operatorname{Irr}_0(b) = \{\mu_1, \dots, \mu_m\}$. By Theorem 4.3, for each i, $\mu_i \in \operatorname{Irr}(N|Q)$ and if $\widetilde{\mu_i} = \Omega_{N,Q}(\mu_i)$, then $\widetilde{\mu_i} \in \operatorname{Irr}_0(\widetilde{b})$.

For $i \in \{1, \dots, m\}$, let Δ_{μ_i} be the natural bijection from $\operatorname{Irr}(G|\mu_i)$ onto $\operatorname{Irr}(N_G(Q)|\widetilde{\mu_i})$ given by Theorem 5.5(i). Next denote by Λ_i, the restriction of Δ_{μ_i} to $\operatorname{Irr}(B|\mu_i)$.

Let $\chi \in \operatorname{Irr}(B|\mu_i)$ ($i \in \{1, \dots, m\}$). Then $\Lambda_i(\chi)$ lies over $\widetilde{\mu_i}$. Since $\widetilde{\mu_i} \in \operatorname{Irr}_0(\widetilde{b})$, it follows by Theorem 5.5(ii) and the Harris–Knörr theorem that $\Lambda_i(\chi) \in \operatorname{Irr}(\widetilde{B}|\operatorname{Irr}_0(\widetilde{b}))$.

Suppose that $\chi \in \operatorname{Irr}(B|\mu_j)$ for some $j \in \{1, \dots, m\}$. Then $\mu_j = \mu_i^x$ for some $x \in G$. We now have $b^x = b$ and so Q^x is a defect group for b. Thus Q^x is N-conjugate to Q and it follows that $x \in NN_G(Q)$. Then μ_j is $N_G(Q)$-conjugate to μ_i, and by Lemma 5.6, we deduce that $\Lambda_j(\chi) = \Lambda_i(\chi)$.

Now the Λ_i's taken together define a map Λ from $\operatorname{Irr}(B|\operatorname{Irr}_0(b))$ to $\operatorname{Irr}(\widetilde{B}|\operatorname{Irr}_0(\widetilde{b}))$. Next we show that Λ is a bijection.

Let $\chi_1 \in \operatorname{Irr}(B|\mu_{i_1})$, $\chi_2 \in \operatorname{Irr}(B|\mu_{i_2})$ and suppose $\Lambda(\chi_1) = \Lambda(\chi_2)$. Then $\Lambda(\chi_1)$ lies over both $\widetilde{\mu_{i_1}}$ and $\widetilde{\mu_{i_2}}$. It follows that $\widetilde{\mu_{i_2}} = \widetilde{\mu_{i_1}}^y$ for some $y \in N_G(Q)$. By Lemma 4.2, we have $\mu_{i_1}^y \in \operatorname{Irr}(N|Q)$ and $\widetilde{\mu_{i_2}} = \Omega_{N,Q}(\mu_{i_1}^y)$. Since $\Omega_{N,Q}$ is an injection (by Theorem 4.1), we get that $\mu_{i_2} = \mu_{i_1}^y$. Now χ_1 and χ_2 both lie over μ_{i_1}, and we have

$$\Lambda_{i_1}(\chi_1) = \Lambda(\chi_1) = \Lambda(\chi_2) = \Lambda_{i_1}(\chi_2).$$

As Λ_{i_1} is one-to-one, we conclude that $\chi_1 = \chi_2$. This shows that Λ is one-to-one.

Suppose now that $\xi \in \operatorname{Irr}(\widetilde{B}|\operatorname{Irr}_0(\widetilde{b}))$. Then, in light of Theorem 4.3, ξ lies over $\widetilde{\mu_k}$ for some $k \in \{1, \dots, m\}$. Next, there exists $\theta \in \operatorname{Irr}(G|\mu_k)$ such that $\xi = \Delta_{\mu_k}(\theta)$. Also, Theorem 5.5(ii) implies that θ belongs to B. Now $\theta \in \operatorname{Irr}(B|\operatorname{Irr}_0(b))$ and

$$\Lambda(\theta) = \Lambda_k(\theta) = \Delta_{\mu_k}(\theta) = \xi.$$

We have thus shown that Λ is onto.

In view of Theorem 5.5(i), every Λ_i preserves character defects. Since B and $\widetilde{B}$ have equal defects, it follows that Λ preserves character heights. The proof of the theorem is now complete. □

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